Gauge–invariant analysis of perturbations in Chaplygin gas unified models of dark matter and dark energy

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Abstract. We exploit the gauge–invariant formalism to analyse the perturbative behaviour of two cosmological models based on the generalized Chaplygin gas describing both dark matter and dark energy in the present Universe. In the first model we consider the generalized Chaplygin gas alone, while in the second one we add a baryon component to it. We extend our analysis also into the parameter range $\alpha > 1$, where the generalized Chaplygin gas sound velocity can be larger than that of light.

In the first model we find that the matter power spectrum is compatible with the observed one only for $\alpha < 10^{-5}$, which makes the generalized Chaplygin gas practically indistinguishable from ΛCDM.

In the second model we study the evolution of inhomogeneities of the baryon component. The theoretical power spectrum is in good agreement with the observed one for almost all values of α. However, the growth of inhomogeneities seems to be particularly favoured either for sufficiently small values of α or for $\alpha \gtrsim 3$. Thus, it appears that the viability of the generalized Chaplygin gas as a cosmological model is stronger when its sound velocity is superluminal. We show that in this case the generalized Chaplygin gas equation of state can be changed in an unobservable region in such a way that its equivalent k-essence microscopical model has no problems with causality.

Keywords: Chaplygin gas, Cosmological perturbation theory, gauge–invariant formalism, large scale structure power spectrum

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1. Introduction

Together with quintessence (a scalar field with some potential minimally coupled to gravity) [1, 2, 3, 4] and other physical and geometrical (modified gravity) models of dark energy (see, e.g., the reviews [5, 6] for exact definitions), the generalized Chaplygin gas [7] (hereafter gCg) is one widely studied model among those proposed to describe the observed accelerated expansion of the universe [8, 9]. In its generalized form, the Chaplygin gas is a barotropic fluid with the following equation of state:

$$p_{\rm Ch} = -A\rho_{\rm Ch}^{-\alpha}, \tag{1}$$

where A and α are positive constants (for the original Chaplygin gas $\alpha = 1$). The generalized equation of state (1) has been introduced in [7] and analysed in [10, 11, 12] and in many other subsequent papers. In contrast to many models describing dark energy alone, the gCg gives a unified description of dark matter and dark energy, enrolling itself in the class of so-called quartessence or unified dark matter (UDM) cosmological models. It allows to interpolate between a dust–dominated phase of the evolution of the Universe in the past and an accelerated one at recent time, see Eq. (2) below. That is why the gCg has attracted much attention in cosmology.

The evolution of the energy density as a function of the cosmic scale factor $a(t)$ of the Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological model is easily obtained from (1) and from the energy conservation equation:

$$\rho_{\rm Ch} = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}, \tag{2}$$

where B is an integration constant usually chosen to be positive (if $B < 0$, the weak energy condition is violated and phantom cosmology takes place, see for example [13, 14]).

From (1) and (2), the parameter $w_{\rm Ch} \equiv p_{\rm Ch}/\rho_{\rm Ch}$ and the square sound velocity $c^2_{\rm sCh}$ have the following expressions:

$$w_{\rm Ch} = -\left[1 + \frac{B}{A}\frac{1}{a^{3(\alpha+1)}}\right]^{-1}, \tag{3a}$$

$$c^2_{\rm sCh} \equiv \frac{{\rm d}p_{\rm Ch}}{{\rm d}\rho_{\rm Ch}} = -\alpha w_{\rm Ch}. \tag{3b}$$

Note that for $\alpha > 1$, $c^2_{\rm sCh}$ may exceed the velocity of light (set to unity in our notations) in the course of the recent or future evolution of the Universe, while it was non-relativistic at large redshifts, during the matter–dominated stage, for all values of α.

Cosmological models based either on one or on more than one fluid are essentially different. The reason for this is that the various fluid components interact indirectly via geometry and may also interact by direct energy exchange. Direct non-gravitational interactions are typically neglected but the indirect ones are always present and can give rise to totally different evolutions, especially at the perturbative level. We will see how this works for the gCg together with baryons.

The pure gCg, namely the cosmological model based on the gCg alone, has passed many tests of standard cosmology. Quite recently, updated constraints for the gCg parameters have been published [15, 16, 17]. However, the behaviour of the gCg under perturbations is still problematic [18, 19]. In [19] the study of the power spectrum of large scale structures seems to indicate that the best fit value of α is very close to zero,

rendering the gCg indistinguishable from ΛCDM [the latter is indeed the limit of gCg for $\alpha \to 0$, as one can infer from (20) below]. This feature has to be attributed to the gCg sound velocity which, during the cosmic evolution, grows from 0 to $\sqrt{\alpha}$ driving inhomogeneities to oscillate (if $\alpha > 0$) or to blow–up (if $\alpha < 0$). This characteristic seems to be common to all UDM models [19] which, as a consequence, might appear to be ruled out.

However, as shown in [20], it appears that adding a baryon component to the gCg the problem disappears and the model is in agreement with observations. But the debate still goes on and the authors of [19] have corroborated their claim about ruling out gCg and, in general, all quartessence models through gravitational lensing measurements on the basis of the current value of the cosmological parameter σ_8 (the rms mass dispersion on a sphere of radius $8h^{-1}$ Mpc).

Still, the parameter σ_8 need not necessarily be a confident discriminant among different cosmological models since it is clearly a quantity which is strongly affected by non–linear effects, while all the calculations performed in [19] are carried out in the linear regime. This point was also raised much earlier in [21] in a more general context.

In the present paper we investigate the issue of cosmological perturbations in gCg models using the gauge–invariant formalism [22]. We consider both the gCg alone and in the presence of baryons, confirming the results of [19] and [20]. In addition, the gCg seems to favour structure formation when its sound velocity is superluminal, in particular when $\alpha \gtrsim 3$.

The structure of the paper is the following:

(i) In section 2 we briefly outline Bardeen gauge–invariant formalism and derive the equations that we will numerically solve in the rest of the paper.

(ii) In section 3 we investigate structure formation in a gCg–dominated universe and display the role of the sound velocity.

(iii) In section 4 we calculate the theoretical power spectrum in a gCg–dominated universe and compare it with the observed one.

(iv) In section 5 we consider a two–fluid model based on gCg in presence of baryons. For this variant of the model we perform the same analysis of the previous two sections and compare the results.

(v) In section 6 we present our conclusions and discuss in more detail why a possible superluminal sound velocity of the gCg is not prohibited by causality arguments.

(vi) For completeness, we add an appendix in which, using the technique developed in [23] we write down the exact solution for perturbations of dust-like matter and generalized Chaplygin gas in the Newtonian approximation at the matter–dominated stage.

2. The Gauge–Invariant Formalism

The cosmological perturbations issue was first tackled and studied by Lifshitz in 1946 in the synchronous gauge [24] (see also [25]). In 1980 Bardeen developed gauge invariant perturbation theory [22] by constructing suitable combinations of metric and stress–energy tensor perturbations which are invariant under a generic gauge transformation. The gauge–invariant formalism has been used and reviewed by many

authors. Here we follow the notation of [26] and study only scalar perturbations. The first–order Einstein equations have the following form:

$$\left\{\begin{array}{l}\Delta\Psi-3\mathcal{H}\left(\mathcal{H}\Phi+\Psi'\right)+3\mathcal{K}\Psi=a^2\delta\rho\\ \mathcal{H}\Phi+\Psi'=a\left(\rho+p\right)V\\ \left[\Psi''+\mathcal{H}\Phi'+\left(2\mathcal{H}'+\mathcal{H}^2\right)\Phi+2\mathcal{H}\Psi'-\mathcal{K}\Psi+\frac{1}{2}\Delta\left(\Phi-\Psi\right)\right]\delta^i_j-\\ -\frac{1}{2}\gamma^{ik}\left(\Phi-\Psi\right)_{|kj}=a^2\delta p\delta^i_j-\sigma^{|i}_{|j},\end{array}\right. \tag{4}$$

where $a(\eta)$ is the scale factor as a function of the conformal time η. Its present value is normalized to unity; $\mathcal{H}(\eta)=a'/a$, where the prime denotes derivation with respect to the conformal time; $\Phi(\boldsymbol{x},\eta)$ and $\Psi(\boldsymbol{x},\eta)$ are the Bardeen gauge–invariant potentials; $\delta\rho(\boldsymbol{x},\eta)$ and $\delta p(\boldsymbol{x},\eta)$ are the gauge–invariant expressions of the perturbations of the energy density and pressure; $V(\boldsymbol{x},\eta)$ is the gauge–invariant expression of the scalar potential of the velocity field; $\sigma(\boldsymbol{x},\eta)$ represents the shear. Finally, γ_{ij} $(i,j=1,2,3)$ is the spatial part of the FLRW metric and $\mathcal{K}=0,\pm1$ is its curvature parameter which corresponds respectively to flat, close and open geometry. The vertical bar denotes covariant derivation with respect to γ_{ij}. Units are chosen such that $4\pi G=c=1$. See [26] for a detailed derivation of system (4).

We introduce two assumptions which simplify (4):

- We assume $\mathcal{K}=0$;
- We neglect shear perturbations, namely we assume $\sigma=0$. This implies $\Phi=\Psi$.

For a cosmological model based on N non–interacting fluids we write for the background quantities:

$$\rho=\sum_{i=1}^N\rho_i,\quad p=\sum_{i=1}^N p_i, \tag{5}$$

while for the perturbations in the linear regime:

$$\delta\rho=\sum_{i=1}^N\delta\rho_i,\quad \delta p=\sum_{i=1}^N\delta p_i. \tag{6}$$

The mixed time–space component of the perturbed energy–momentum tensor will look like $\sum_{i=1}^N\left(\rho_i+p_i\right)V_i$, since each fluid will contribute with its own velocity.

Adiabatic perturbations of a barotropic fluid are characterized by the following equation of state:

$$\delta p_i=c^2_{\mathrm{s}i}\delta\rho_i, \tag{7}$$

where

$$c^2_{\mathrm{s}i}\equiv\frac{\partial p_i}{\partial\rho_i} \tag{8}$$

is the sound velocity at constant entropy for the generic component i. In the cosmological perturbation theory it is customary to deal with the density contrast:

$$\delta_i\equiv\frac{\delta\rho_i}{\rho_i}, \tag{9}$$

where ρ_i is the background energy density of the component i. Since we work in the linear regime it is useful to take the spatial Fourier transform of (4) which allows us to treat each mode independently. With the above assumptions and trading the conformal time for the scale factor a, system (4) for a generic multi–fluid model is rewritten as follows:

$$\left\{\begin{array}{l}-k^2\Phi-3a\mathcal{H}^2\dot\Phi-3\mathcal{H}^2\Phi=a^2\sum_{i=1}^N\rho_i\delta_i\\ \mathcal{H}\Phi+a\mathcal{H}\dot\Phi=a\sum_{i=1}^N\left(\rho_i+p_i\right)V_i\\ \left(a\mathcal{H}\right)^2\ddot\Phi+\left(4a\mathcal{H}^2+a^2\mathcal{H}\dot{\mathcal{H}}\right)\dot\Phi+\left(2a\mathcal{H}\dot{\mathcal{H}}+\mathcal{H}^2\right)\Phi=a^2\sum_{i=1}^N c^2_{\mathrm{s}i}\rho_i\delta_i,\end{array}\right. \tag{10}$$

where the dot denotes derivation with respect to the scale factor a (for the sake of simplicity we adopt the same notation for the Fourier transformed quantities as for the original ones).

If $N = 1$, the system is determined and can be solved by eliminating δ and by extracting the following second order equation for Φ:

$$\ddot{\Phi} + \left(\frac{\dot{\mathcal{H}}}{\mathcal{H}} + \frac{4}{a} + \frac{3c_{\rm s}^2}{a}\right)\dot{\Phi} + \left(2\frac{\dot{\mathcal{H}}}{a\mathcal{H}} + \frac{1+3c_{\rm s}^2}{a^2} + \frac{k^2 c_{\rm s}^2}{a^2\mathcal{H}^2}\right)\Phi = 0. \tag{11}$$

Once solved, we can use the solution of (11) in the first equation of (10) to find the solution for δ.

Instead, if $N > 1$ we need $2(N-1)$ additional equations to make (10) determined. These equations should take into account the possible interactions and the energy exchanges among the different fluids. Since we have assumed that they are not mutually interacting directly, the required additional equations to be added can be simply chosen as the energy conservation equations, $\delta T^{\mu}_{\nu;\mu} = 0$, given separately for each further component. Namely the continuity equation $\delta T^{\mu}_{0;\mu} = 0$:

$$\dot{\delta}_i + \frac{3}{a}\left(c_{{\rm s}i}^2 - w_i\right)\delta_i - 3\dot{\Phi}\left(1 + w_i\right) + \frac{k^2}{a^2\mathcal{H}}\left(1 + w_i\right)V_i = 0 \tag{12}$$

and the Euler equation $\delta T^{\mu}_{l;\mu} = 0$:

$$\left[\left(\rho_i + p_i\right)V_i\right]^{\cdot} + \frac{3}{a}\left(\rho_i + p_i\right)V_i - \frac{c_{{\rm s}i}^2\rho_i}{\mathcal{H}}\delta_i - \frac{\left(\rho_i + p_i\right)}{\mathcal{H}}\Phi = 0, \tag{13}$$

where $w_i \equiv p_i/\rho_i$. Both (12) and (13) are Fourier transformed, that is why there is no l subscript in (13).

It is possible to write the combination of (10), (12) and (13) as a system of N second order equations, one for each δ_i. One needs to derive (12), then to express $\ddot{\Phi}$, $\dot{\Phi}$, V_i and $\dot{V}_i$ in terms of the other δ_i's through (10), (13) and again (12). In the case $N = 2$ the resulting system has the following form:

$$\begin{cases} \ddot{\delta}_1 + \left(\frac{\dot{\mathcal{H}}}{\mathcal{H}} + \frac{A_1}{\mathcal{H}}\right)\dot{\delta}_1 + \frac{B_1}{\mathcal{H}}\dot{\delta}_2 + \frac{C_1}{\mathcal{H}^2}\delta_1 + \frac{D_1}{\mathcal{H}^2}\delta_2 = 0 \\ \\ \ddot{\delta}_2 + \left(\frac{\dot{\mathcal{H}}}{\mathcal{H}} + \frac{A_2}{\mathcal{H}}\right)\dot{\delta}_2 + \frac{B_2}{\mathcal{H}}\dot{\delta}_1 + \frac{C_2}{\mathcal{H}^2}\delta_2 + \frac{D_2}{\mathcal{H}^2}\delta_1 = 0, \end{cases} \tag{14}$$

where the coefficients have the following expressions:

$$\begin{aligned} A_1 = {} & 2\frac{\mathcal{H}}{a} + 3\frac{\mathcal{H}}{a}\left(c_{\rm s1}^2 - 2w_1\right) \\ & -3a\mathcal{H}\left(\rho_1 + p_1\right)\frac{3\mathcal{H}^2\left(3c_{\rm s1}^2 - 1\right) + k^2\left(3c_{\rm s1}^2 + 1\right) + 6\left(\mathcal{H}^2 - \mathcal{H}\dot{\mathcal{H}}a\right)}{k^4 + \left(3\mathcal{H}^2 + k^2\right)\left(\mathcal{H}\dot{\mathcal{H}}a - \mathcal{H}^2\right)}, \end{aligned} \tag{15}$$

$$B_1 = -3a\mathcal{H}\rho_2\left(1 + w_1\right)\frac{3\mathcal{H}^2\left(3c_{\rm s1}^2 - 1\right) + k^2\left(3c_{\rm s1}^2 + 1\right) + 6\left(\mathcal{H}^2 - \mathcal{H}\dot{\mathcal{H}}a\right)}{k^4 + \left(3\mathcal{H}^2 + k^2\right)\left(\mathcal{H}\dot{\mathcal{H}}a - \mathcal{H}^2\right)}, \tag{16}$$

$$\begin{aligned} C_1 = {} & \frac{k^2 c_{\rm s1}^2}{a^2} + \frac{3}{a^2}\left(\dot{\mathcal{H}}\mathcal{H}a + 4\mathcal{H}^2\right)\left(c_{\rm s1}^2 - w_1\right) + 3\frac{\mathcal{H}^2}{a}\left(c_{\rm s1}^2\right)^{\cdot} - 3\left(1 + w_1\right)c_{\rm s1}^2\rho_1 + \\ & 3\mathcal{H}^2\left(2 + 3c_{\rm s1}^2\right)\left(\rho_1 + p_1\right)\frac{k^2 - 3\left(c_{\rm s1}^2 - w_1\right)\left(3\mathcal{H}^2 + k^2\right)}{k^4 + 3\left(3\mathcal{H}^2 + k^2\right)\left(\mathcal{H}\dot{\mathcal{H}}a - \mathcal{H}^2\right)} \end{aligned}$$

$$-\left(k^2+3\mathcal{H}^2-6\mathcal{H}\dot{\mathcal{H}}a\right)(\rho_1+p_1)\,\frac{k^2+3\left[\mathcal{H}\dot{\mathcal{H}}a-\mathcal{H}^2-3\mathcal{H}^2\left(c_{\rm s1}^2-w_1\right)\right]}{k^4+3\left(3\mathcal{H}^2+k^2\right)\left(\mathcal{H}\dot{\mathcal{H}}a-\mathcal{H}^2\right)}, \tag{17}$$

$$D_1=-3\left(1+w_1\right)c_{\rm s2}^2\rho_2+$$
$$3\mathcal{H}^2\left(2+3c_{\rm s1}^2\right)\rho_2\left(1+w_1\right)\frac{k^2-3\left(c_{\rm s2}^2-w_2\right)\left(3\mathcal{H}^2+k^2\right)}{k^4+3\left(3\mathcal{H}^2+k^2\right)\left(\mathcal{H}\dot{\mathcal{H}}a-\mathcal{H}^2\right)}$$
$$-\left(k^2+3\mathcal{H}^2-6\mathcal{H}\dot{\mathcal{H}}a\right)\rho_2\left(1+w_1\right)\frac{k^2+3\left[\mathcal{H}\dot{\mathcal{H}}a-\mathcal{H}^2-3\mathcal{H}^2\left(c_{\rm s2}^2-w_2\right)\right]}{k^4+3\left(3\mathcal{H}^2+k^2\right)\left(\mathcal{H}\dot{\mathcal{H}}a-\mathcal{H}^2\right)}. \tag{18}$$

The coefficients A_2, B_2, C_2 and D_2 have the same form, but with the interchange $1\leftrightarrow 2$ in the subscripts.

As shown in [23], at the matter–dominated regime, i.e. when $p_1\ll\rho_1$, $p_2\ll\rho_2$ and $c_{\rm s1},c_{\rm s2}\ll 1$, the above coefficients are far simpler and system (14) can be rewritten in the following quasi–Newtonian form:

$$\begin{cases}\ddot{\delta}_1+\left(\frac{\dot{\mathcal{H}}}{\mathcal{H}}+\frac{2}{a}\right)\dot{\delta}_1+\frac{k^2}{a^2\mathcal{H}^2}c_{\rm s1}^2\delta_1=\frac{1}{\mathcal{H}^2}\left(\rho_1\delta_1+\rho_2\delta_2\right)\\[2ex] \ddot{\delta}_2+\left(\frac{\dot{\mathcal{H}}}{\mathcal{H}}+\frac{2}{a}\right)\dot{\delta}_2+\frac{k^2}{a^2\mathcal{H}^2}c_{\rm s2}^2\delta_2=\frac{1}{\mathcal{H}^2}\left(\rho_1\delta_1+\rho_2\delta_2\right).\end{cases} \tag{19}$$

This approximation is sufficient to study the formation of all gravitationally bound objects (galaxies, in particular) which become nonlinear for $z>1$. As pointed out in [23], it is possible to recover the system (19) by considering the limit $k\gg\mathcal{H}$, too, but it is not a necessary constraint in the matter–dominated regime when $c_{\rm s1}$ and $c_{\rm s2}$ are small. It is interesting that the system (19) can be exactly solved for the two–fluid model considered in Sec. 5. The solution is exhibited in the Appendix.

3. Perturbations in a Chaplygin gas dominated universe

In the gCg dominated universe the Hubble parameter scales as follows:

$$\frac{\mathcal{H}^2}{\mathcal{H}_0^2}=\left(\bar{A}+\frac{1-\bar{A}}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}a^2, \tag{20}$$

where $\bar{A}=A/(A+B)$ and $\mathcal{H}_0=H_0$ is the Hubble constant.

In a typical analysis of the gCg model it is customary to take $\alpha<1$ since the square sound velocity tends to α for $a\to\infty$, as can be seen from ($3b$). For $\alpha=1$ the sound velocity will tend to unity in the far future, at $t\to\infty$. The new regime that we investigate in the present paper is $\alpha>1$, thereby making gCg superluminal after a sufficiently long but still finite time.

From ($3b$) it is straightforward to calculate at which redshift the transition to the superluminal gCg would occur. For generic values of $\bar{A}$ and α:

$$z_{\rm s}=\left[\frac{\bar{A}\left(\alpha-1\right)}{1-\bar{A}}\right]^{\frac{1}{3(\alpha+1)}}-1. \tag{21}$$

Given $\alpha>1$ and $\bar{A}$, it is plausible to expect that some, hopefully observable, cosmological effect would take place at that redshift.

It is crucial to point out that α and $\bar{A}$ are not independent, but linked by (20), once we have some constraints about the background expansion. In the present paper

we consider z_{tr}, i.e. the redshift at which the transition to the accelerated phase of the expansion takes place. We derive its expression from (20):

$$z_{\mathrm{tr}} = \left[\frac{2\bar{A}}{1-\bar{A}}\right]^{\frac{1}{3(\alpha+1)}} - 1. \tag{22}$$

It follows from the most recent SN Ia observations combined with CMB and baryon acoustic oscillations (BAO) data that $z_{\mathrm{tr}} \approx 0.7$ with a rather large uncertainty (at least ± 0.1 or more at the 2σ confidence level), see e.g. Fig. 7 in [27] and also [17, 28, 29]. For comparison, if only SN Ia are used, then the Gold+HST dataset [30] leads to $z_{\mathrm{tr}} \sim 0.4$, see Fig. 2 in [27] and also [31]) while other datasets (SNLS and ESSENCE) alone produce values of $z_{\mathrm{tr}} \sim 0.7$, too (see [17, 27]). However, these numbers strongly depend on the present value of the non-relativistic matter density Ω_{m0} which, in turn, has to be expressed through $\bar{A}$ in the gCg case (the values given above correspond to $\Omega_{\mathrm{m0}} \approx 0.28$). Direct calculation of z_{tr} for UDM models [16, 28] shifts this quantity to even larger redshifts: $z_{\mathrm{tr}} = 0.8 - 0.9$. Actually, in the latter case, the authors consider only silence quartessence (namely the effective sound velocity is assumed to be zero) and the parameter α is constrained to be $\alpha = -0.06 \pm 0.1$.

Then from (22) we can write down an explicit relation between $\bar{A}$ and α:

$$\bar{A} = \frac{(1+z_{\mathrm{tr}})^{3(1+\alpha)}}{2 + (1+z_{\mathrm{tr}})^{3(1+\alpha)}}. \tag{23}$$

Note that the uncertainty $\sigma_{z_{tr}}$ of the transition redshift propagates on $\bar{A}$ through the following formula:

$$\sigma_{\bar{A}} = \frac{6(\alpha+1)}{[2(1+z_{\mathrm{tr}})^{-3(\alpha+1)} + 1]^2 (1+z_{\mathrm{tr}})^{3\alpha+4}} \sigma_{z_{tr}}, \tag{24}$$

therefore the larger is α, the smaller is the bias on $\bar{A}$.

A widely used cosmological parameter stemming from CMB observation is R, the comoving distance to the last scattering surface scaled to Ω_{m0}, namely:

$$R = H_0 \sqrt{\Omega_{\mathrm{m0}}} \int_0^{z_{\mathrm{ls}}} \frac{dz}{H(z)}, \tag{25}$$

where $z_{\mathrm{ls}} \approx 1089$ is the last scattering surface redshift. In UDM models, the quantity Ω_{m0} should be taken from the asymptotic behaviour of $\mathcal{H}^2$ for $z \gg 1$ at the matter–dominated stage, so $\Omega_{\mathrm{m0}} = \left(1-\bar{A}\right)^{1/(\alpha+1)}$ (or $\Omega_{\mathrm{m0}} = \left(1-\bar{A}\right)^{1/(\alpha+1)} (1-\Omega_b) + \Omega_b$ if baryons are taken into account, too). At the 1σ confidence level, $R = 1.71 \pm 0.03$ (see for example [32]).

An interesting limiting case is the 'super–duperluminal' one: $\alpha \to \infty$. Then $H \equiv \mathcal{H}/a = \mathrm{const} = H_0$ for $z \le z_{\mathrm{tr}}$ and $H^2(z) = H_0^2 \left((1+z)/(1+z_{\mathrm{tr}})\right)^3$ for larger z. In this case, one can obtain an analytic expression for R:

$$R_\infty = \frac{z_{\mathrm{tr}}}{(1+z_{\mathrm{tr}})^{3/2}} + \frac{2}{\sqrt{1+z_{\mathrm{tr}}}} - \frac{2}{\sqrt{1+z_{\mathrm{ls}}}} \; . \tag{26}$$

Then the above mentioned observational window for R is reached for $z_{\mathrm{tr}} = 1.0 \pm 0.1$.

Combining (22) with (21) we find that

$$z_{\mathrm{s}} = \left(\frac{\alpha-1}{2}\right)^{\frac{1}{3(\alpha+1)}} (1+z_{\mathrm{tr}}) - 1. \tag{27}$$

If $\alpha = 3$, then $z_{\mathrm{s}} = z_{\mathrm{tr}}$. For $\alpha > 3$, $z_{\mathrm{s}} > z_{\mathrm{tr}}$ but z_{s} approaches z_{tr} once more for $\alpha \to \infty$. For a fixed z_{tr}, the maximal value of z_{s} is reached for $\alpha \approx 8.182$ when

$(1 + z_{\mathrm{s}})/(1 + z_{\mathrm{tr}}) \approx 1.048$. Thus, for $\alpha \gtrsim 3$, the transition from subluminal to superluminal gCg occurs approximately at the same time as the transition to the accelerated phase of expansion of the Universe.

We now compare the evolution of perturbations in the gCg–dominated universe for different values of α. We choose an integration range which starts at the decoupling era, namely $z \sim 1100$ ($a \approx 10^{-3}$), and ends up when the first structures, namely protogalaxies, were formed ($a \approx 0.1$, or $z \approx 10$), so that the linear approximation holds true.

We numerically solve (11) choosing $[-1, 0]$ as normalized initial conditions for $\left[\Phi, \dot{\Phi}\right]$. In our calculations we use (23) to properly choose $\bar{A}$ as a function of α. We show the results using $z_{tr} = 0.8 \pm 0.16$ and $z_{tr} = 1.0 \pm 0.1$.

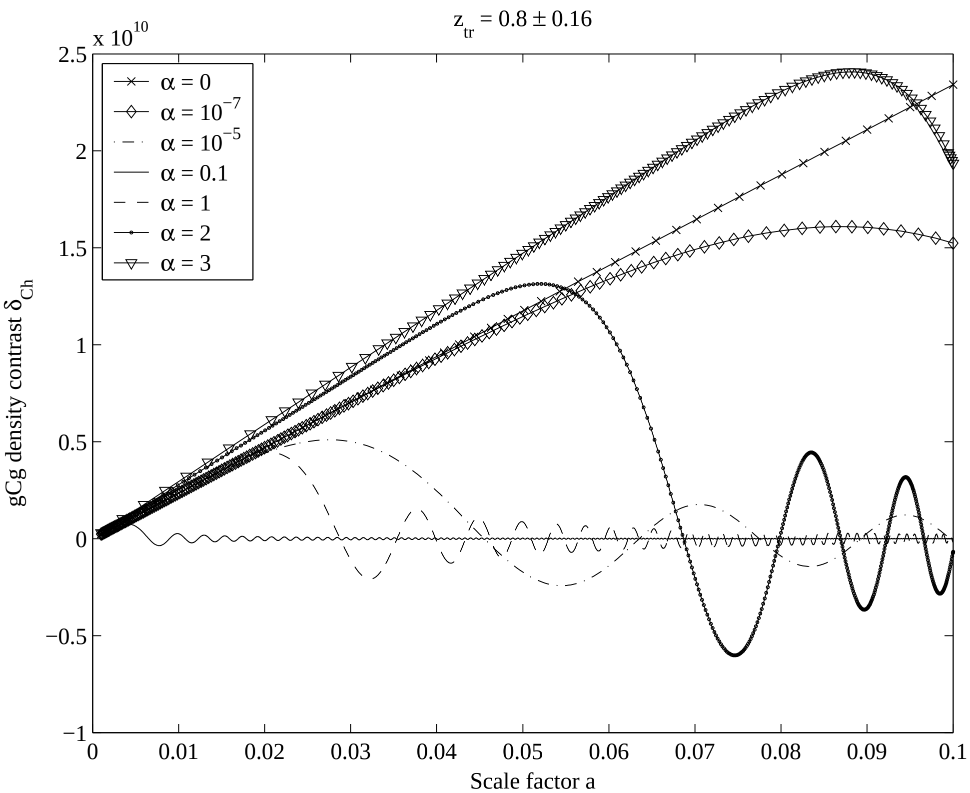

Figure 1. Evolution profiles of δ_{Ch} for different values of α and for $k = 100$ h Mpc^{-1}. The transition redshift is $z_{tr} = 0.8 \pm 0.16$

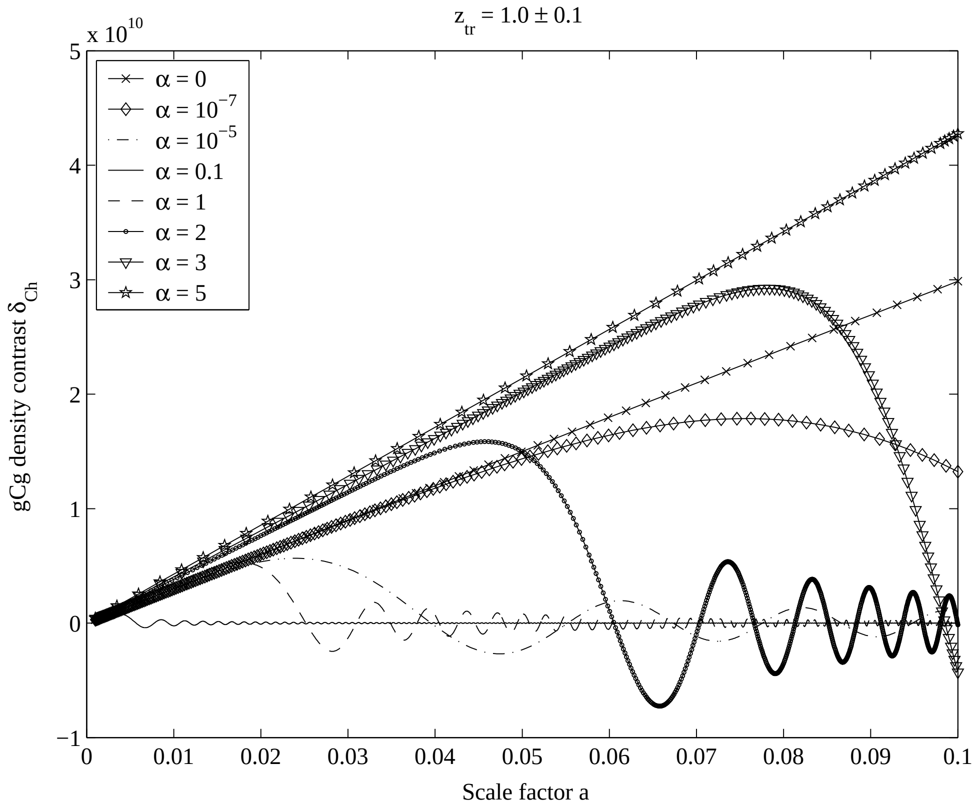

Figure 2. Same as figure 1, with $z_{tr} = 1.0 \pm 0.1$

The plots in figures 1 and 2 display the evolution of the density contrast in the gCg, which we call $\delta_{\rm Ch}$, for different choices of the parameter α and for $k = 100$ h Mpc^{-1}, which corresponds to a scale of order 50 h^{-1} *kpc*, which is typical of a protogalaxy. The case $\alpha = 0$ corresponds to the ΛCDM model.

Indeed, here we are first investigating if formation of gravitationally bound objects including galaxies takes place at all. That is why we have chosen so small a scale and we have stopped at $z = 10$. The gCg UDM model, like all CDM–like ones, belongs to the class of bottom–up, or hierarchical clustering, scenarios of structure formation where small scale structures are formed first. The question of large–scale structure in this model at scales which remain linear for $z < 1$, of course, requires integration up to $z = 0$. It is considered in the next section.

Part of the results found in [19] are already confirmed by this analysis, without a deeper study involving the power spectrum. The development of oscillations takes place too early for $\alpha \gtrsim 10^{-5}$, thus preventing structure formation.

However, two further interesting features can be extracted from figures 1 and 2. First, it seems that there exists a critical value, namely $\alpha \approx 0.1$, for which the deviation of the gCg from the $\alpha = 0$ linear growth is maximal. Second, the larger the value of α (in the range $\alpha > 0.1$) the smaller the deviation from the $\alpha = 0$ behaviour. Indeed, when $\alpha > 1$, i.e. in the superluminal regime, the oscillations are absent and

the growth seems even to be enhanced.

Both the above mentioned features can be explained by the behaviour of the gCg sound velocity as a function of α, which is displayed in figure 3 for $z_{tr} = 0.8 \pm 0.16$ (the plots for $z_{tr} = 1.0 \pm 0.1$ are very similar).

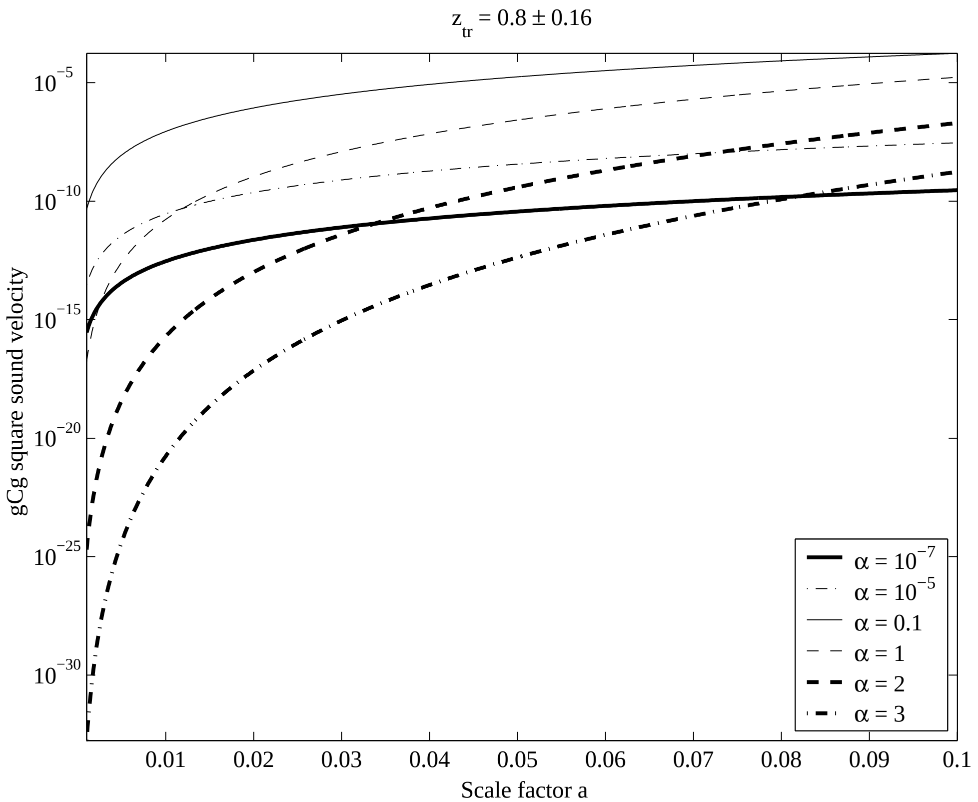

Figure 3. The gCg square sound velocity given in ($3b$) plotted as a function of a for different values of α. $\bar{A}$ is given by (23) with $z_{tr} = 0.8 \pm 0.16$.

For $\alpha \sim 0.1$ the sound velocity becomes non negligible much earlier than at other values of α. The range $10^{-7} \lesssim \alpha \lesssim 3$ appears thus to be ruled out since structure formation is prevented. However, to understand what happens out of this range it is necessary to study the clustering properties of matter at the present time, namely the power spectrum. This will be done in the following section.

4. Large Scale Power spectrum of Chaplygin gas perturbations

In the previous section we have shown that in the gCg model galaxy formation is possible only when α is very small or $\alpha \gtrsim 3$. In this section we try and constrain these bounds further by studying the clustering properties of gCg perturbations, namely the power spectrum $P(k)$, and comparing it with observation.

We exploit the SDSS (Sloan Digital Sky Survey) data as analysed in [33]. The sample considered by the authors consists of 205,443 galaxies observed before July 2002. The data consist of RA (Right Ascension), Dec (Declination) and redshift z for each galaxy. The redshift is converted to comoving distance by using a flat cosmological model with a cosmological constant $\Omega_\Lambda = 0.7$ (it can be shown that our results are robust to this assumption).

The power spectrum is computed at present time, namely at $z = 0$. The data processed in [33] are displayed in table 1.

Table 1. Effective k, window functions and measured values of $P(k)$ with relative bias factors and standard deviations. $[k] = hMpc^{-1}$ and $[P(k)] = (h^{-1}Mpc)^3$. From the SDSS data reported in [33]. We have neglected two rows of data in which σ is greater than the respective $P(k)$.

k	**Low k**	**High k**	$P(k)$	**Bias**	σ
0.01819	0.01502	0.02440	33254.63896	1.16745	24572.79357
0.02405	0.01984	0.03136	38360.58264	1.16674	13320.48227
0.02776	0.02300	0.03571	24143.07699	1.16613	10047.21709
0.03201	0.02663	0.04048	19709.29306	1.16532	7413.82995
0.03691	0.03094	0.04601	12595.77528	1.16428	5485.84481
0.04257	0.03591	0.05215	13558.60084	1.16294	4077.68223
0.04912	0.04162	0.05983	18311.08054	1.16124	2974.15048
0.05670	0.04824	0.06873	12080.73574	1.15910	2140.39514
0.06527	0.05561	0.07867	9217.46084	1.15647	1580.01728
0.07529	0.06455	0.08999	9750.49986	1.15317	1127.95803
0.08698	0.07507	0.10351	9529.63935	1.14906	818.49019
0.10037	0.08670	0.11898	6384.82545	1.14409	601.70752
0.11581	0.09985	0.13679	5294.92081	1.13813	446.65015
0.13360	0.11478	0.15748	4629.83669	1.13109	335.07950
0.15412	0.13181	0.18139	3573.92724	1.12290	254.13547
0.17774	0.15114	0.20891	3393.82814	1.11358	195.23431
0.20489	0.17288	0.24062	2298.10000	1.10320	152.73819
0.23592	0.19690	0.27653	1597.14976	1.09193	123.50981
0.27062	0.22219	0.31395	1105.42903	1.08019	107.20069
0.30618	0.23123	0.34782	1012.72489	1.06912	110.33632

We compute the gCg power spectrum for different values of α, by first solving (11), thus finding the transfer function and then applying it to the following prior:

$$|\delta_k|^2 = Nk \left[\frac{\ln(1+2.34q)}{2.34q}\right]^2 \left[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4\right]^{-1/2}, \quad (28)$$

where N is a normalization constant and

$$q \equiv \frac{k}{\Omega_{X0}h^2}, \quad (29)$$

where Ω_{X0} is the cold dark matter (CDM) energy density fraction evaluated today, and h is the Hubble constant in units of 100 $km\ s^{-1}\ Mpc^{-1}$. In our calculations we

shall set $h = 0.7$. Since the gCg has the same asymptotic behaviour as the ΛCDM model in the matter–dominated stage, then the cold dark matter fraction is given by

$$\Omega_{X0} = \left(1 - \bar{A}\right)^{\frac{1}{1+\alpha}} . \tag{30}$$

The prior (28) is obtained by applying the BBKS transfer function [34] to the scale invariant Harrison–Zel'dovich spectrum. The normalization constant N is computed through a best fit of the data for each α. In our calculation, $\bar{A}$ is constrained by (23) using $z_{tr} = 0.8 \pm 0.16$ and $z_{tr} = 1.0 \pm 0.1$.

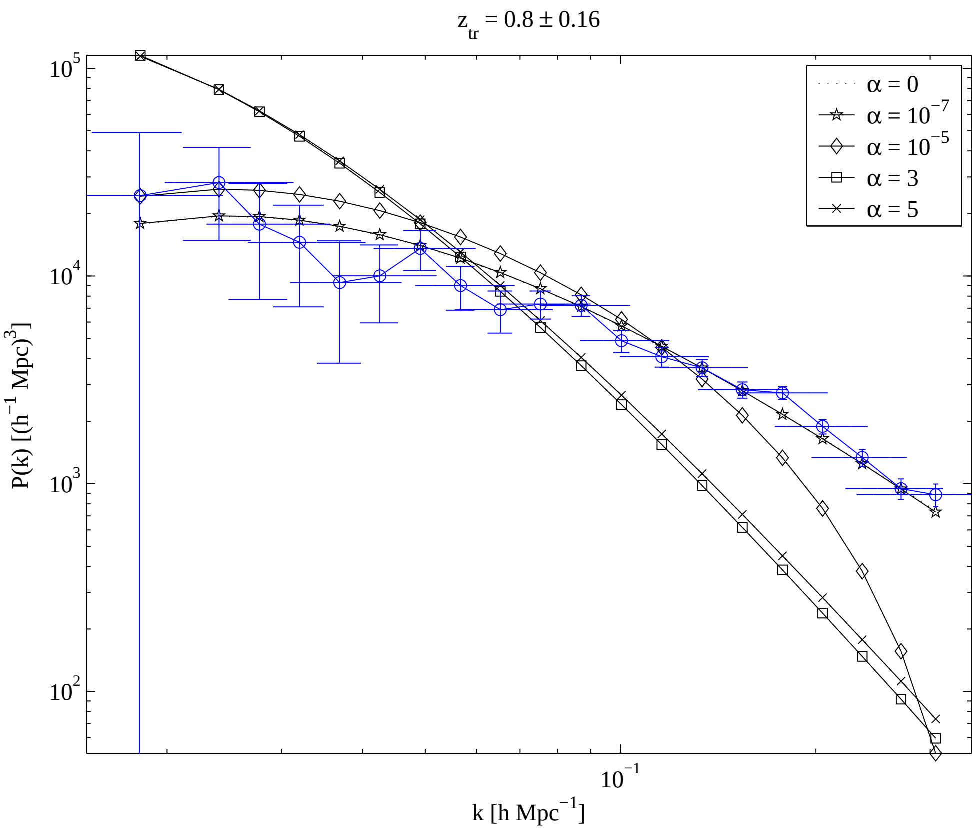

Figure 4. Theoretical power spectra for different values of α compared with the measured one. $\bar{A}$ is given by (23) with $z_{tr} = 0.8 \pm 0.16$. Note that the plots for $\alpha = 0$ and $\alpha = 10^{-7}$ are superposed. We have extracted the envelopes of the oscillations of the density contrasts for $\alpha = 3$ and $\alpha = 5$ and have used them for the calculations. For the other cases this procedure was unnecessary since no oscillations were present.

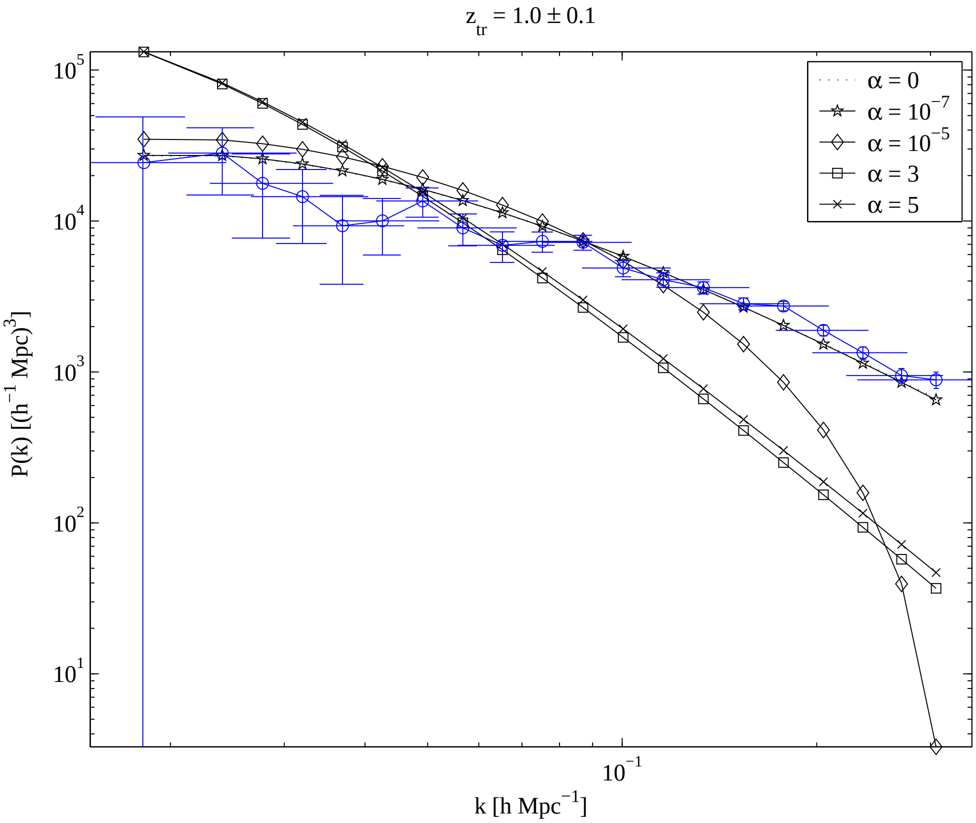

Figure 5. Same as figure 4, with $z_{tr} = 1.0 \pm 0.1$.

In figures 4 and 5 the power spectra have been computed by considering the envelope of the oscillations of the density contrast δ_{Ch} and by using it in the subsequent calculations. This procedure serves to clearly compare the different slopes of the calculated power spectra with respect to the observed one. Moreover, it has been carried out only for $\alpha = 3$ and $\alpha = 5$. For the other ones it was unnecessary as no oscillations were present.

An interesting feature that can be drawn from figures 4 and 5 is that the larger is α the more the power spectrum tends to a limiting behaviour which is systematically below that of the ΛCDM one. This happens because the suppression of the gCg transfer function relative to that of the ΛCDM model behaves approximately as k^{-1} for large values of α after averaging over oscillations (actually, the exponent is slightly larger than -1 because of the accelerated expansion at recent times).

Figures 6 and 7 show what happens if the oscillations in the density contrasts are taken into account. The power spectra for $\alpha = 3$ and $\alpha = 5$ are highly irregular and their slopes are hardly recognizable.

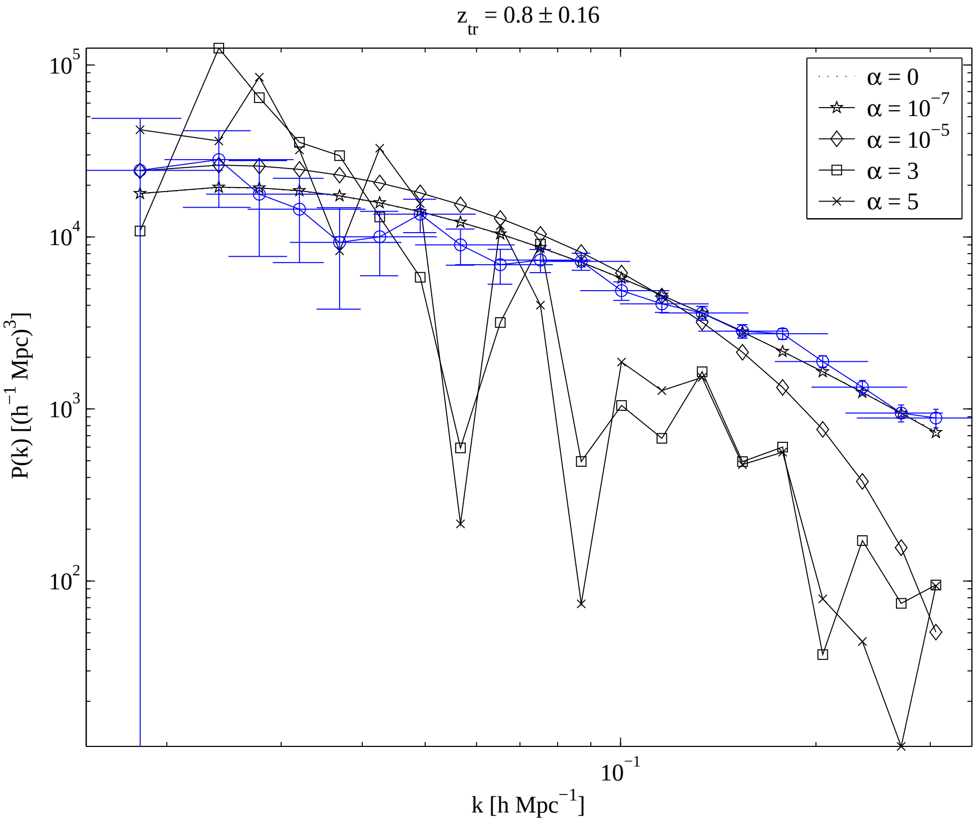

Figure 6. Theoretical power spectra for different values of α compared with the measured one. $\bar{A}$ is given by (23) with $z_{tr} = 0.8 \pm 0.16$. Note that the plots for $\alpha = 0$ and $\alpha = 10^{-7}$ are superposed.

In conclusion, the pure gCg does not seem to work from a perturbative viewpoint, even when it is superluminal. However, a reasonable possibility is to allow the gCg to have non–adiabatic perturbations, which is a natural assumption since it is not a pressureless fluid. An attempt in this direction has already been performed in [35] and [36] (here also a baryon component is considered). However, the results are based on the *ad hoc* assumption of silent perturbations (namely $\delta p_{\rm Ch} = 0$), which seems questionable.

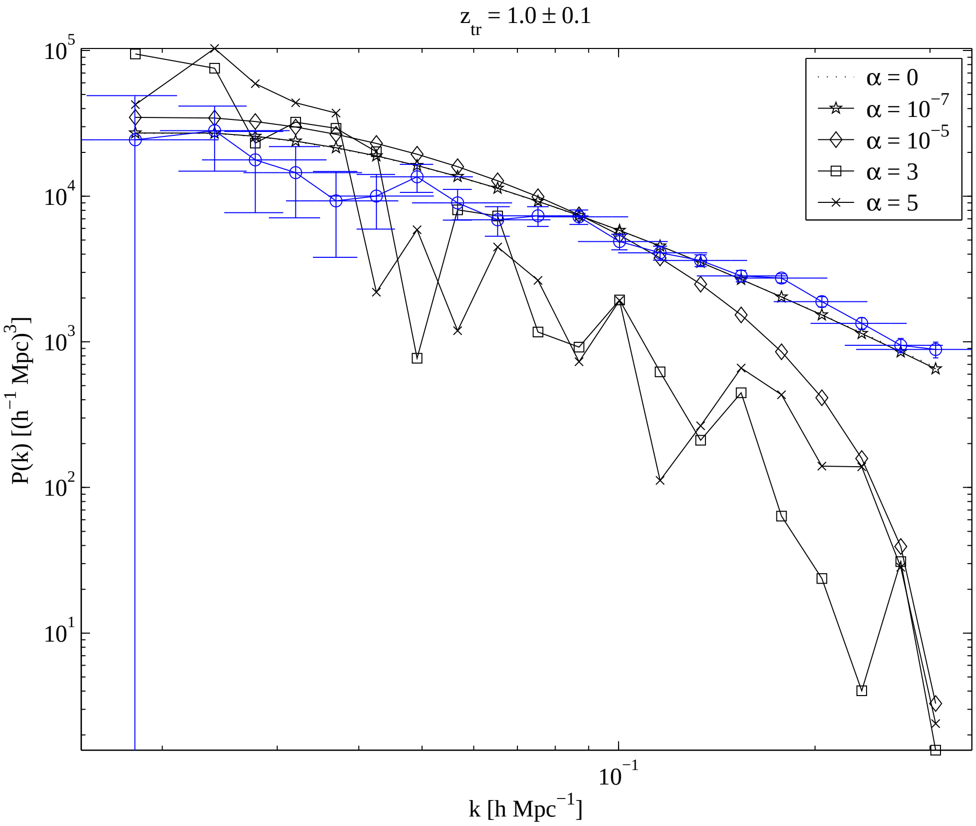

Figure 7. Same as figure 6, with $z_{tr} = 1.0 \pm 0.1$.

5. Two-fluid model: generalized Chaplygin gas plus baryons

In this section we consider a two fluid model including baryons together with gCg. The Friedmann equation governing this model has the following form:

$$\frac{\mathcal{H}^2}{\mathcal{H}_0^2} = \left\{ \frac{\Omega_{\rm b0}}{a^3} + (1 - \Omega_{\rm b0}) \left[\bar{A} + \frac{1 - \bar{A}}{a^{3(\alpha+1)}} \right]^{1/1+\alpha} \right\} a^2, \tag{31}$$

where subscript b refers to the baryonic component and subscript 0 indicates that the corresponding quantity is evaluated at the present epoch ($a = 1$).

System (14) is solved numerically in the same integration range of the previous sections and setting $\Omega_{\rm b0} = 0.04$. Before proceeding it is necessary to find a relation between $\bar{A}$ and α which ensures the preservation of the correct background expansion, e.g. the acceleration redshift $z_{\rm tr} \approx 0.6$.

From (31) it is possible to work out the following expression:

$$\frac{\bar{A}\left[(1 + z_{\rm tr})^{-3(\alpha+1)} + 1/2\right] - 1/2}{\left\{\bar{A}\left[(1 + z_{\rm tr})^{-3(\alpha+1)} - 1\right] + 1\right\}^{\alpha/\alpha+1}} = \frac{\Omega_{\rm b0}}{2\,(1 - \Omega_{\rm b0})}, \tag{32}$$

which cannot be inverted to give $\bar{A}$ as a function of α. However, since $\Omega_{\rm b0}$, is small we can use again (23) to that purpose.

As in the previous section, we choose (28) as a prior for the power spectrum, but since this time we have the presence of baryons, Ω_{X0} has to be chosen differently. We exploit then Sugiyama's shape correction [37]:

$$\Omega_{X0} = \left[\Omega_{b0} + (1-\Omega_{b0})\left(1-\bar{A}\right)^{\frac{1}{1+\alpha}}\right] \times \\ \exp\left(-\Omega_{b0} - \frac{\sqrt{2h}\Omega_{b0}}{\Omega_{b0} + (1-\Omega_{b0})\left(1-\bar{A}\right)^{\frac{1}{1+\alpha}}}\right), \quad (33)$$

which properly takes into account the presence of baryons.

In the appendix we show an exact solution, in the matter dominated regime, for the evolution of perturbations in the model of the present section.

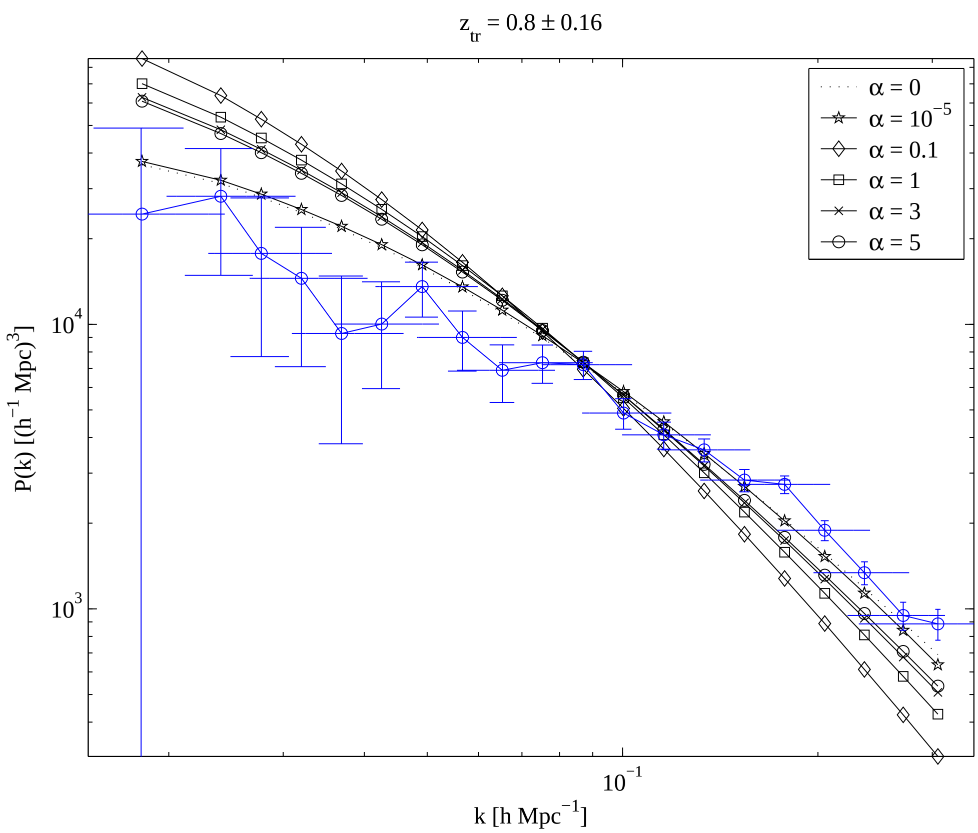

Figure 8. Power spectra of the baryonic component computed for different values of α with $z_{tr} = 0.8 \pm 0.16$.

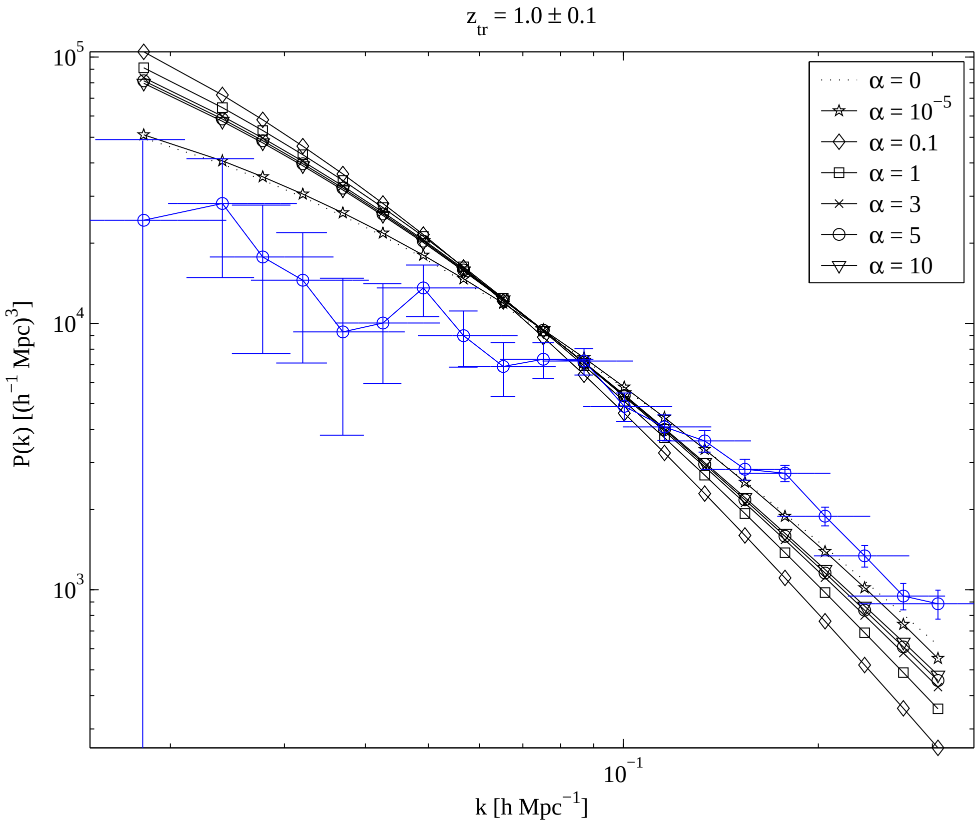

Figure 9. Same as figure 8 with $z_{tr} = 1.0 \pm 0.1$.

In figures 8 and 9 we show the baryonic power spectra computed for different values of α compared to the observed one. We have chosen as normalized initial conditions the following values: $\left[\delta_{\rm b}, \dot{\delta}_{\rm b}, \delta_{\rm Ch}, \dot{\delta}_{\rm Ch}\right] = [1, 1, 1, 1]$. We have chosen the same initial conditions for all the quantities essentially because after the equivalence era the gCg behaved as dust and therefore its density contrast grew linearly (as a function of the scale factor).

Unlike what happened in the previous section, this time there were no oscillations in δ_b, since the baryonic density contrast is unaffected by the instabilities of the gCg one. In fact, the results are different from those inferred from figures 4 and 5. The agreement with the observed power spectrum is better. It is particularly good again for very small α but also in the range $\alpha > 1$. Therefore the gCg seems to work better either when it is practically indistinguishable from ΛCDM or when its sound velocity is superluminal.

The good agreement of the theoretical power spectrum with the observed one can be explained realizing that gCg perturbations are indistinguishable from those of baryons at the beginning of the matter–dominated stage. Therefore they really act as dark matter and compel baryons perturbations to grow and form structures. On the other hand, when the gCg sound velocity increases, the growth of the baryons

inhomogeneities is damped, but does not oscillate. This fact can be observed in figures 10 and 11. Moreover, structures which have already formed are not influenced by the gCg sound velocity, so neither is the large scale power spectrum. This was already pointed out in [20].

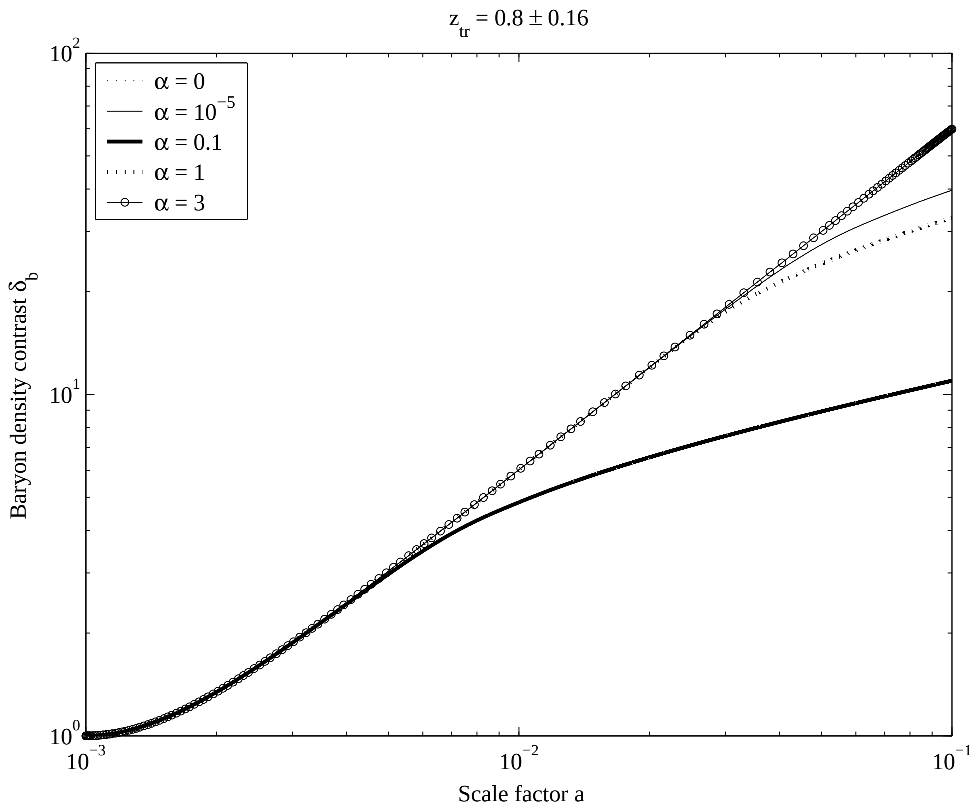

Figure 10. Evolution profiles of $\delta_{\rm b}$ for different values of α and for $k = 100\ h$ Mpc^{-1}. Note that the plot for $\alpha = 0$ and $\alpha = 3$ are superposed. $z_{tr} = 0.8 \pm 0.16$.

In figures 10 and 11 we display the evolution of $\delta_{\rm b}$ for different values of α choosing again $\left[\delta_{\rm b}, \dot{\delta}_{\rm b}, \delta_{\rm Ch}, \dot{\delta}_{\rm Ch}\right] = [1, 1, 1, 1]$ as normalized initial conditions. As in section 3 we have chosen $k = 100\ h\ Mpc^{-1}$. Note that the plots corresponding to $\alpha = 0$ and $\alpha = 3$ are superposed.

We infer from figures 10 and 11 that the gCg sound velocity has a damping effect on the growth of baryon inhomogeneities. These are not compelled to oscillate, but structure formation is delayed, if not prevented. In order to render structure formation as similar as possible to that of ΛCDM, we need either very small values of α or $\alpha > 3$. The latter range for α just leads to $z_{\rm s} > z_{\rm tr}$ as was shown in section 3.

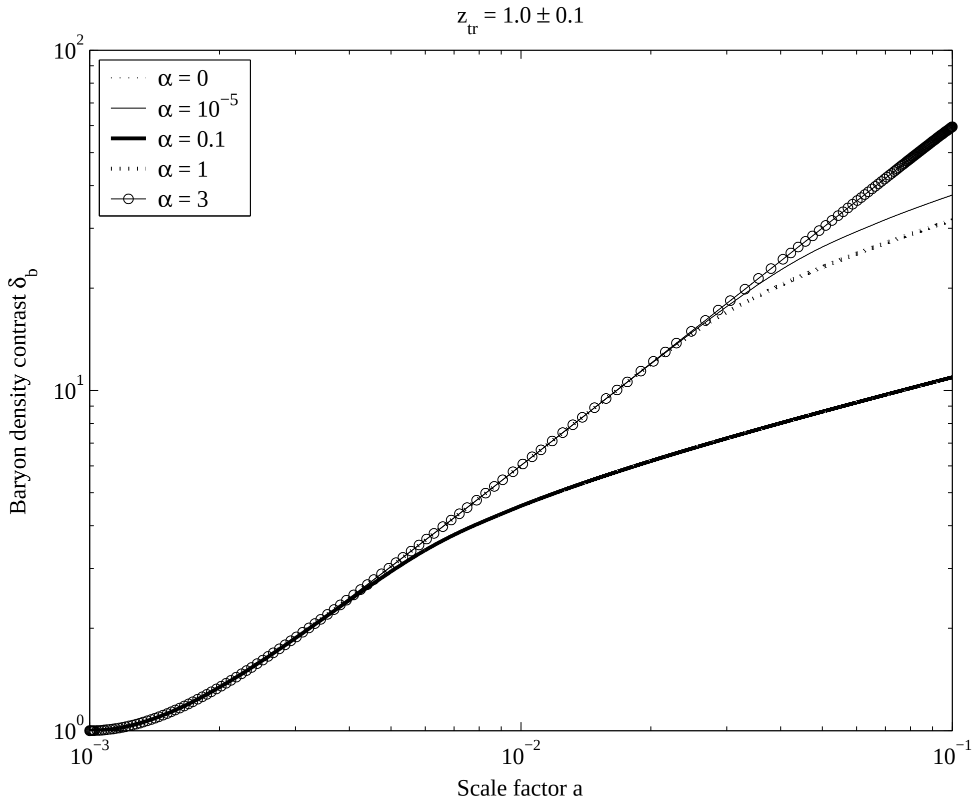

Figure 11. Same as figure 10, with $z_{tr} = 1.0 \pm 0.1$.

6. Conclusions

From our analysis it emerges that:

- The generalized Chaplygin gas cosmological model, with no additional fluid components, is compatible with structure formation and large scale structure clustering properties (encoded in the power spectrum) only for α sufficiently small ($\alpha < 10^{-5}$), in which case it is practically indistinguishable from the ΛCDM model.
- Adding to the generalized Chaplygin gas model a baryon component we find that the growth of the density contrast and the large scale structure clustering properties are compatible with observations for all values of α. However very small values of α and the case $\alpha \gtrsim 3$ are favoured.

Note, however, that the integrated Sachs-Wolfe (ISW) effect (not considered in this paper) leads to rather large enhancement of low CMB multipoles C_l in UDM models, in contrast to a modest increase of C_l in the ΛCDM model [44], that results in the exclusion of the gCg with $0.01 < \alpha \leq 1$ [18, 45, 46] (see also the recent paper [47] for an analytical treatment of the ISW effect for UDM models). It may be interesting to investigate if it remains so large for all $\alpha > 1$.

Now, since most of the new results of this paper refer to the superluminal case $\alpha > 1$, it is natural to consider the question if this range is physically admissible in more detail. The fact that the group velocity $c_{\rm sCh}$ in media may exceed the light velocity in vacuum (unity in our notations) is not, in itself, unphysical. Not only it is not prohibited from the theoretical point of view, but this phenomenon has been actually observed in laboratories in case of light propagation through dispersive media. As was proved long ago (see e.g. [38]), what is really necessary in order not to violate causality is that the signal (or, the wavefront) velocity should not exceed 1.

Then, what is the signal velocity for the gCg? It does not appear possible to answer this question at the hydrodynamic level since the gCg phenomenological equation of state in the normal (non-phantom) case requires $\rho_{\rm Ch} \geq A^{1/(1+\alpha)}$, so that the limit $\rho_{\rm Ch} \to 0$ corresponding to a wavefront expanding into vacuum cannot be studied. Therefore, to determine the value of the signal velocity for the gCg, one has to use some underlying microscopic field–theoretical model from which the equation of state (1) arises in macroscopic hydrodynamics.

We will use the so-called tachyon representation of the gCg for this purpose where the gCg is described by a scalar field ϕ (k-essence) with the Lagrangian density [11]

$$\mathcal{L} \equiv p_{\rm Ch} = -V_0 \left[1 - \left(\frac{X}{V_0}\right)^{\frac{1+\alpha}{2\alpha}}\right]^{\frac{\alpha}{1+\alpha}} \quad , \quad X = \frac{1}{2}\phi_{,\mu}\phi^{,\mu} < V_0 \ , \tag{34}$$

$$\rho_{\rm Ch} = 2X\frac{d\mathcal{L}}{dX} - \mathcal{L} = V_0 \left[1 - \left(\frac{X}{V_0}\right)^{\frac{1+\alpha}{2\alpha}}\right]^{-\frac{1}{1+\alpha}} \quad , \quad A = V_0^{1+\alpha} \ . \tag{35}$$

Though other microscopic models are also possible (see e.g. discussion in [39] in the case $\alpha = 1$), the present one has the following advantages: a) its solutions are in one–to–one correspondence to those of the hydrodynamic gCg if $X > 0$, and b) it has the same equations for perturbations, too.

Now we are able to consider the limit $X \to 0$ corresponding to a wavefront expanding into vacuum with no k-essence in front of it. To prove that the signal velocity is equal to unity in this case, it is sufficient to show that the wave equation corresponding to (34), namely

$$\mathcal{L}_X \phi^{;\mu}_{;\mu} + \mathcal{L}_{XX}\phi^{,\mu}\phi^{,\nu}\phi_{;\mu;\nu} = 0, \tag{36}$$

has plane-wave solutions of the type $f(x-t)$ in flat space-time (since the signal velocity is also the infinite momentum limit of the phase velocity of a wave). It is straightforward to check that the sufficient condition for this to hold is that both $\mathcal{L}_X$ and $\mathcal{L}_{XX}$ are finite at $X = 0$ (a similar remark is made in the recent paper [40], see also [41]). In addition, $\mathcal{L}_X(X = 0)$ should be non-negative in order for ghosts not to be present.

In the superluminal case $\alpha > 1$, the Lagrangian density (34) does not satisfy this condition since $\mathcal{L}_X$ and $\mathcal{L}_{XX}$ diverge at $X \to 0$. However, without going into a more detailed analysis of the singularity structure at a null hyper–surface in this case, we propose a simple *sufficient* way to cure the causality problem for the gCg. Namely, let us assume that the expression (34) changes to

$$\mathcal{L} = -V_0 + X\left(\frac{V_0}{V_1}\right)^{\frac{\alpha-1}{2\alpha}} \tag{37}$$

for $X \ll V_1$ where $V_1 \ll V_0$ and that it smoothly matches (34) at $X \sim V_1$. Then it is clear that the signal velocity is exactly unity. The Lagrangian density (37) describes a massless scalar field minimally coupled to gravity and a cosmological constant.

The correspondingly corrected macroscopic gCg remains the same in the observable region $X \gtrsim V_0$, or $\rho_{\rm Ch} + p_{\rm Ch} \gtrsim V_0$ (note that V_0 is of the order of the present day critical energy density in the Universe) but will become a mixture of a cosmological constant and the extremely stiff ideal fluid ($p = \rho$) in the far future when X will drop to V_1. With the same accuracy, one can say that the corrected gCg becomes the usual Chaplygin gas when its density is very close to V_0, much closer than the one which occurs presently. Thus, by slightly changing the gCg equation of state in the unobservable region, the causality problem is cured. As a result, the present gCg superluminal sound velocity is a *transient* phenomenon which disappears at $t \to \infty$.

Finally, note that in the limit $\alpha \to \infty$, the gGg realizes the Parker-Raval scenario [42] of a sudden transition from the matter–dominated stage with $a(t) \propto t^{2/3}$ to the de Sitter expansion with a constant curvature (modulo a small term $\propto \Omega_b$). The original idea of Parker and Raval to produce such a scenario using non-perturbative vacuum polarization of a light minimally coupled scalar field is questionable, both regarding derivation of the effective action and stability of the resulting specific kind of the $f(R)$ theory of gravity (here R is the Ricci scalar). But the same scenario based on the limiting case of the gCg does not have these problems. This shows once more that the superluminal gCg deserves further, more detailed study.

Acknowledgments

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Appendix A. Exact solution for perturbations of the gCg+baryons model at the matter–dominated regime

At the matter–dominated regime, the system (19) describing gCg and baryons takes the following form:

$$\left\{ \begin{array}{l} \ddot{\delta}_{\rm b} + \left(\frac{\dot{\mathcal{H}}}{\mathcal{H}} + \frac{2}{a}\right)\dot{\delta}_{\rm b} = \frac{1}{\mathcal{H}^2}\left(\rho_b \delta_{\rm b} + \rho_{Ch}\delta_{\rm Ch}\right) \\ \ddot{\delta}_{\rm Ch} + \left(\frac{\dot{\mathcal{H}}}{\mathcal{H}} + \frac{2}{a}\right)\dot{\delta}_{\rm Ch} + \frac{k^2}{a^2\mathcal{H}^2}c_{\rm s}^2\delta_{\rm Ch} = \frac{1}{\mathcal{H}^2}\left(\rho_b \delta_{\rm b} + \rho_{Ch}\delta_{\rm Ch}\right), \end{array} \right. \tag{A.1}$$

where $c_{\rm s}^2$ is the gCg square sound velocity given by $(3a)$ and $(3b)$.

This system can be exactly solved using the technique developed in [23] for cosmological perturbations of two–fluid models in the Newtonian approximation. Let us introduce the following variables:

$$x = k\gamma^{-2}a^{-\frac{3}{2}\gamma}, \quad \gamma = -2\alpha - \tfrac{8}{3}, \tag{A.2}$$

where α is the gCg parameter. It is possible to extract from system (A.1) a fourth order equation for $\delta_{\rm Ch}$:

$$\left[\left(\Delta + \frac{2}{3\gamma}\right)\Delta\left(\Delta - \frac{1}{3\gamma}\right)\left(\Delta - \frac{1}{\gamma}\right) + \right.$$

$$x\left(\Delta^2+\frac{2\gamma-1/3}{\gamma}\Delta+\frac{1}{\gamma^2}\left(\gamma\left(\gamma-\frac{1}{3}\right)-\frac{2}{3}\Omega_{\rm b0}\right)\right)\Bigg]\,\delta_{\rm Ch}=0, \quad \text{(A.3)}$$

where $\Delta = x\cdot \mathrm{d}/\mathrm{d}x$.

As found in [23], the general solution of (A.3) can be represented in terms of the Meijer G–functions (the generalized hypergeometric functions):

$$\begin{aligned}\delta_{\rm Ch} = {} & C_1 G^{41}_{24}\left(x|^{a_1 a_2}_{b_1 b_2 b_3 b_4}\right) + C_2 G^{41}_{24}\left(x|^{a_2 a_1}_{b_1 b_2 b_3 b_4}\right) + \\ & + C_3 G^{40}_{24}\left(x\mathrm{e}^{\mathrm{i}\pi}|^{a_1 a_2}_{b_1 b_2 b_3 b_4}\right) + C_4 G^{40}_{24}\left(x\mathrm{e}^{-\mathrm{i}\pi}|^{a_1 a_2}_{b_1 b_2 b_3 b_4}\right),\end{aligned} \quad \text{(A.4)}$$

where:

$$\begin{array}{ll} a_{1,2}=\frac{1}{6\gamma}\left(1\mp\sqrt{1+24\Omega_{\rm b0}}\right) & \\ b_1=-\frac{2}{3\gamma}, & b_2=0 \\ b_3=\frac{1}{3\gamma}, & b_4=\frac{1}{\gamma}, \end{array} \quad \text{(A.5)}$$

the C_i $(i=1,2,3,4)$ are integration constants and the Meijer G–functions are defined as follows [43]:

$$G^{m,n}_{p,q}\left(x|^{a_1,\dots,a_p}_{b_1,\dots,b_q}\right)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma_L}\frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=n+1}^p\Gamma(a_j-s)\prod_{j=m+1}^q\Gamma(1-b_j+s)}x^s ds, \quad \text{(A.6)}$$

where $\Gamma(s)$ is the Gamma function and the integration contour $\gamma_{\rm L}$ circuits all the poles of $\Gamma(1-a_j+s)$ anti–clockwise and all the poles of $\Gamma(b_j-s)$ clockwise (all these poles lie along the real axis).

The solution for $\delta_{\rm b}$ has the same functional form as (A.4), with:

$$a_{1,2}=1+\frac{1}{6\gamma}\left(1\mp\sqrt{1+24\Omega_{\rm Ch0}}\right), \quad \text{(A.7)}$$

where $\Omega_{\rm Ch0}=1-\Omega_{\rm b0}$ since we assume a spatially flat universe.

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